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WEIGHTED E – OPTIMALITY CRITERIA FOR COMPARISON OF BALANCE INCOMPLETE BLOCK DESIGN

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ABSTRACT

This paper investigates conditions under which balanced incomplete block designs enjoy weighted optimality with E-criterion establishing weighted intervals for E–optimal design. More so, a neighbourhood of weights for grouped generalised divisible designs (GGDDs) maintaining E_w – optimal in D (v, b, k) was also investigated. The E-criterion was shown to be closely related to efficiency balance. Bounding arguments that are important tools in tackling E-optimality problems was employed; the standard bounds were generalized for seeking E-weighted optimal (E_W-optimal) designs. The optimal bound established the best conceivable values of the criterion and thus the designs with these best values are optimal.

KEYWORDS: Weighted Optimality, Incomplete Block Design, Group Generalised Divisible Design, Bounding Arguments, Weighted Intervals

INTRODUCTION

In the design of experiments, optimal designs are a class of experimental designs that are optimal with respect to some statistical criterion. According to Ipinyomi, (2012), under the design of experiments for estimating statistical models, optimal designs allow parameters to be estimated without bias and with minimum variance. A non – optimal design requires a greater number of experimental runs to estimate the parameter with the same precision as an optimal design. In practical terms, optimal experiments can reduce the costs of experimentation. The optimality of a design depends on the statistical model and is assessed with respect to a statistical criterion, which is related to the variance matrix of the estimator. Specifying an appropriate model and specifying a suitable criterion function both require understanding of statistical theory and practical knowledge with designing experiments.

Gupta et al (1999) and Gupta et al (2002) used the term "weighted optimality" when comparing a group of test treatments with a group of control treatments. Two different sets of control, treatment – control and treatment – treatment were considered to be estimated with unequal precision. Design of experiments for which some of the treatments are controls has form a special perspective, been extensively investigated in recent years. Notable among the many papers seeking optimal designs for test treatment versus control experiments are Jacroux (1989), Jacroux and Majundar (1989), Majundar (1992, 1996), Majundar and Notz (1983) and Stutken (1991). Optimality work for T v C experiment is a limiting case, as the weight on the control treatment goes to 1.

Kiefer (1975) introduced convex optimality function ϕ on the information matrices and proved that balance incomplete block designs (BIBDs) are universally optimal, i.e. minimise ϕ (Cd) for every non – increasing, convex permutation – invariant ϕ. Following closely on the heels of Kiefer's work, John and Mitchell (1977), Cheng (1978, 1980) and Jacroux (1980) used computer search and theoretical arguments to build optimal designs for the criteria defined above, all of which fall into Kiefer's frame work. More recently, Majundar and Notz (1983), Majundar (1986), Jacroux and Majundar (1989), Bagchi and Shah (1989), Bagchi and Berkum (1991), Bagchi (1996), Bagchi and Bagchi (2001), Reck and Morgan (2005) and Morgan (2000, 2003, 2007) have been working on design optimality for various classes of designs with blocking.

Let Y_{uj} be the observation on experimental unit u in block j. The commonly employed statistical model for any block design d, which in many cases is justifiable by randomization alone (Hinkelmann and Kempthorne, 2008) is

$$Y_{uj} = \mu + \tau_{d(u,j)} + \beta_j + \epsilon_{uj}, \tag{1}$$

Where

μ = mean response over all treatment and blocks,

$_{d(u,j)}$ = the treatment assigned to unit u in block j by design d,

$\tau_{d(u,j)}$ = the effect of the treatment assigned to unit u in block j by design d,

β_j = the effect of block j,

and the ϵ_{uj} 's are uncorrelated, mean zero random variables with common variance σ_E^2. This model is employed in most of the papers cited above. It is the basis for the information matrix mentioned earlier on. We usually assumed with no loss of generality that the unit variability σ_E^2 is $\sigma_E^2 = 1$. The symbol n is used for the total number of experimental units, n = bk

Weighted Optimality of Block Designs

Kiefer's design optimality is based on functions of the information matrix that are invariant to treatment permutation, that is $\phi\ (PC_dP^T) = \phi\ (C_d)$ for any permutation matrix P. This implies equal interest in all treatments. However, in practice there are many cases where not all treatments are equally important. For instance, we often encounter experimental situations where some test treatments are to be compared to a standard treatment (or control treatment). Sometimes the control is included specifically to verify the expectation of large treatment effects relative to control, after which the important comparisons among test treatments are performed. This indicates asymmetry of interest on test treatments and the control treatment with (in this case) greater interest in test treatments than control. Asymmetry of treatment interest implies that optimality based on the information matrix should not be invariant to all permutations. According to Wang (2009), with the premise of asymmetric interest, the approach here is to group treatments into several subsets which are assigned distinct weights; larger weight reflects greater interest placed on estimating comparisons involving the corresponding treatments. In situations like that described above, this leads to a 2 – weight design problem, that is, the weights take only two values, with one small weight and v-1 larger, equal weights.

Definition 1

Let positive weights $w_1, w_2, \ldots w_v$ be measures of interest on v treatments without loss of generality $\sum_{i=1}^{v-1} w_i = 1$. Let W be a v x v diagonal matrix with w_i in the i^{th} diagonal position, that is, W = Diag (w_i). Also, the square root matrix for W is denoted as $W^{½}$. Then the weighted information matrix C_{dw} for design d is defined as

$C_{dw} = W^{½}\, C_d\, W^{½} = ((c_{dii'}/ \sqrt{w_i\, w_{i'}}))$ (2)

The use of C_{dw} will be justified in the following facts. The key is to see how applying weights to C_d induces weights on variances of treatment contrasts. Consider the spectral decomposition of the C_{dw} –matrix:

$C_{dw} = W^{-½}\, C_d W^{-½} = \sum_{i=1}^{v-1} \theta_i f_i f'_i$. (3)

Where $\theta_0 < \theta_1 \leq . . . \leq \theta_{v-1}$ are the eigenvalues of C_{dw} and the f_i are an orthonormal set of eigenvectors. For connected designs, both C_d and C_{dw} are of rank v – 1, and $\theta_0 = 0$. Let $w = (w_1, w_2, . . . , w_v)'$ be the vector of weights. The eigenvector corresponding to θ_0 is $f_0 = w^{½} = (\sqrt{w_1}, \sqrt{w_2}, . . . , \sqrt{w_v},)'$

Basics for the Weighted *E*- Criterion

Definition 2

The weighted *E* – value (written as E_w) for a design *d* is the largest canonical weighted variance for design *d*. That is,

$E_w = \frac{1}{\theta_1}$ (4)

A design $\bar{d}$ is weighted *E*-optimal (or E_w-optimal) in a design class *D* if it minimises the largest canonical weighted variance, that is, if

$E_{\bar{d}w} = \min_{d \in D} E_{dw}$ (5)

Result 2: The weighted *E*-value is the largest weighted variance over all treatment contrasts.

Proof 2: The largest weighted variance over all contrasts is

$$\max_{c'1=0} \left(\frac{\mathrm{Var}\,(\widehat{c'\tau})}{c'W^{-1}c}\right) = \max_{c'1=0} \left(\frac{c'W^{-1/2}C^{+}_{dw}W^{-1/2}c}{c'W^{-1}c}\right) = \max_{y'w^{1/2}=0} \left(\frac{y'C^{+}_{dw}\,y}{y'y}\right) \quad (6)$$

$w^{1/2}$ is an eigenvector of C^{+}_{dw} corresponding to eigenvalue 0, so this is the largest eigenvalue of C^{+}_{dw}, that is $1/\theta_1$.

Result says that an E_w-optimal design factors importance of contrasts into design selection in minimising impact of the worst case. It can be seen that for any design, increasing the weight placed on a treatment increases weighted variances $\mathrm{Var}_w\,(\widehat{c'\tau}) = [c'W^{-1}c]^{-1}\,\mathrm{Var}\,(\widehat{c'\tau})$ of contrasts in which it is involved. Minimising summary functions of weighted variances (that is, minimising functions of $1/\theta_i$), pushes variances of treatments with higher weight to be smaller, this being at the expense of variances of treatments with smaller weights.

STATEMENT OF THE PROBLEM

The general formulation on the studies of optimality for treatment comparison is based on the idea that optimality functions of the treatment information matrix are invariant to treatment permutation which implies equal interest in all treatments. In practice, however, there are many experiments where not all treatments are equally important. When selecting a design for such an experiment, it would be better to weigh the information gathered on different treatments according to their relative importance and/or interest. It is on this premise that this research work is based on weighted optimality criteria.

AIM AND OBJECTIVES

The broad objective of this research is to explore weighted optimality of incomplete block designs. The specific objectives are:

- To establish weighted intervals for E –weighted optimal (E_w – optimal) design covering possible Treatment with Control (T_wC) situation having smaller weight on contrast.
- A neighbourhood of weights for Grouped Generalised Divisible Designs (GGDD) maintaining E_w - optimality in D (v, b, k) is also investigated.

MATERIALS AND METHODS

The conventional criteria for evaluating design optimality are functions of the eigenvalues of the information matrix C_d, in the same way, many of the weighted criteria which is used to evaluate design optimality are function of the weighted information matrix C_{dw} . There are many statistical packages for analysing the information matrix of a design optimality such as MINITAB, SPSS and R – Statistics. In this research, I made use of R – Statistical package.

ANALYSIS OF RESULTS

This chapter deals with the analysis and interpretation of data. I considered different designs of different sizes for E_w –optimality criteria and a neighbourhood of weight for grouped generalised divisible designs (GGDD).

Analysis and Results of Optimal Block Designs

Example 1: For $v_1 = 6$, $v_2 = 3$, $w_1 = 1/12$ and $w_2 = 1/6$, the following design is E_w – optimality over D (v, b, k) = (9, 11, 6)

1	1	1	1	1	2	2	2	3	3	1
2	2	3	4	5	3	4	5	4	4	2
3	4	5	6	6	6	5	6	5	6	3
7	7	7	7	7	7	7	7	7	7	4
8	8	8	8	8	8	8	8	8	8	5
9	9	9	9	9	9	9	9	9	9	6

It can be observed that the above design is built up by adding the treatment 7, 8 and 9 to every block in BIBD (6, 10, 3), then appending symbols (1, . . . , 6).

Example 2: The following design in D (7, 17, 4) is E_w – optimal for $v_1 = \{1, 2, 3\}$, $v_2 = \{4, 5, 6, 7\}$, $w_1 = 1/4$ and $w_2 = 1/16$

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	5
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	6
4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7	7

Grouped Generalised Divisible Design

Example 3: The following two designs are built up from BIBD (7, 7, 3). The first design adds $\hat{b}$ = 1 copy of first block in BIBD (7, 7, 3) to obtain a GGDD (2). The second one adds $\hat{b}$ = 2 copies of the first block in BIBD (7, 7, 3).

For $w_1 = 2$, $w_2 = 1$, $v_1 = 3$ and $v_2 = 4$, this design is E_w – optimal in D (v, b, k) = (7, 8, 3).

1	1	1	2	2	3	3	1
2	4	6	4	5	4	5	2
3	5	7	6	7	7	6	3

For $w_1 = 3/13$, $w_2 = 1/13$, $v_1 = 3$ and $v_2 = 4$, this design is E_w – optimal in D (v, b, k)

1	1	1	2	2	3	3	1	1
2	4	6	4	5	4	5	2	2
3	5	7	6	7	7	6	3	3

Example 4: Consider the following BIBD (5, 10, 3)

1	1	1	1	1	1	2	2	2	3
2	2	2	3	3	4	3	3	4	4
3	4	5	4	5	5	4	5	5	5

$k/v = 3/5$, observed that $2/3 > k/v > 1/2$, so a = 1 (by theorem). It can be checked that $\lambda (v - k - 1) < 2 (k - 1)$, so $\hat{d}$ can be any positive number. The designs constructed by appending $\hat{b}$ copies of one block in the above BIBD are E_w – optimal in D (v, b, k) = (5, 10 +$\hat{b}$, 3) for $v_1 = 3$, $v_2 = 2$ and $w_1/w_2 = (\lambda + \hat{b})/\lambda = b/3 + 1$.

Table 1: Parameters of Weight Balanced, Binary Block Designs

V	(v_1, v_2)	k	b	r_1	r_2	λ_{11}	λ_{22}	λ_{12}	w_1:w_2	Design #
5	(2, 3)	3	10	9	4	9	1	3	3:1	1
			19	15	9	12	3	6	2.1	2
			20	18	8	18	2	6	3:1	3
			30	27	12	27	3	9	3:1	4
6	(2, 4)	3	26	21	9	18	2	6	3:1	5
		4	13	12	7	12	3	6	2:1	6
			26	24	14	24	6	12	2:1	7
			27	22	16	18	8	12	3:2	8
	(3, 3)	3	11	7	4	4	1	2	2:1	9
			22	14	8	8	2	4	2.1	10
			29	22	7	16	1	4	4:1	11
7	(2, 5)	3	22	18	6	16	1	4	4:1	12
		4	17	14	8	12	3	6	2:1	13
	(3, 4)	3	17	5	9	1	4	2	1:2	14
			23	15	6	9	1	3	3:1	15
		4	17	16	5	16	1	4	4:1	16
			21	16	9	12	3	6	2:1	17
			23	20	8	18	2	6	3:1	18
	(2, 6)	4	30	27	11	27	3	9	3:1	19
8	(3, 5)	4	26	18	10	12	3	6	2:1	20
	(4, 4)	4	18	13	5	9	1	3	3:1	21
			29	17	12	9	4	6	3:2	22

Table 1:Contd.,

9	(2, 7)	3	24	15	6	9	1	3	3:1	23
		4	24	20	8	18	2	6	3:1	24
	(3, 6)	6	21	20	11	20	5	10	2:1	25
	(4, 5)	4	26	6	16	1	9	3	1:3	26
		5	26	25	6	25	1	5	5:1	27
10	(2, 8)	4	18	16	5	16	1	4	(4:1)	28
	(3, 7)	3	25	11	6	4	1	2	2:1	29
		5	15	3	7	8	2	4	2:1	30
	(5, 5)	5	20	13	7	8	2	4	2:1	31
12	(2, 10)	4	19	13	5	9	1	3	3:1	32

CONCLUSIONS

Weighted intervals for Ew – optimality was established and it was shown to be closely related to efficiency balance. More so, bounding arguments establishes the best conceivable value of E-criterion in the first main result for Ew-optimal ruling out classes of inferior designs and since each of these designs achieved the best conceivable value respectively and the E_w – optimal designs for the 2 – weight problem with group sizes ≥ 2 i.e. GGDD(2), the necessary and sufficient condition for d to have weighted information matrix $C_{dw} = Є (I – w^{1/2} w^{1/2\prime})$ for some Є and some w_1 and w_2 are met. Hence the designs considered in this paper are good designs.

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